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Density correlations of magnetic impurities and disorder

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Abstract. We consider an electron coupled to a random distribution of point vortices in the plane (magnetic impurities), with a spatial probability distribution governed by Bose or Fermi statistics at a given temperature. We analyse the effect of the statistics of the magnetic impurities on the partition function and the density of states (DOS) of a test particle. Comparison is made with the Poisson distribution, i.e. Bose or Fermi statistics at infinite temperature. We show in particular that, for the zero-temperature Fermi distribution, the DOS always exhibits oscillations, whatever the strength of the magnetic disorder, contrarily to the Poissonian case. A diagram describing isolated impurities versus Landau level oscillations is proposed.

1. Introduction

Recently, the problem of a two-dimensional electron gas coupled to a static random magnetic field has been a subject of interest [1–5]. Particular attention has been paid to localization properties of such systems. In the case of Gaussian disorder with zero mean, all states seem to be localized [2]. In contrast, they are delocalized in the case of a uniform magnetic field. Therefore, the question arises about the role played by a mean-field description of a random magnetic field [3, 4].

In [2], on the one hand, a Gaussian-disordered magnetic field was considered, on top of a constant homogeneous magnetic field. A semiclassical analysis indicated that the broadening of the Landau levels increased with the energy, so that the DOS oscillations get exponentially damped. In [5], on the other hand, a Harper model was studied, with a random flux per plaquette. Similar results were obtained via numerical simulations, with a resulting flattening of the DOS with increasing disorder.

We are presently interested in a two-dimensional model for an electron of charge e (test particle) coupled to a random magnetic field [4] consisting of a random distribution of infinitely thin vortices, with an average density ρ. Each vortex carries a flux ϕ and modelizes some sort of magnetic impurity, characterized by the dimensionless Aharonov–Bohm coupling $\alpha = e\phi/2\pi$ (i.e. ϕ in unit of the quantum of flux, $\hbar = 1$). The model is periodic in α with period 1 and since there is no privileged orientation of the plane, it is invariant by changing α into $-\alpha$, implying that α can be restricted to the interval $[0, \frac{1}{2}]$.

Random magnetic impurity systems are different from [2, 5] because the magnetic field is generated by a random distribution of point fluxes, rather than by a continuum random field or random fluxes through the plaquettes of a lattice. In order to study these systems, one may perturbatively evaluate in α the average one-electron partition function. It gives information on the average DOS, since it is its Laplace transform.

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In [4], we focused on magnetic impurities obeying a Poisson distribution. The main conclusions were:

• the DOS is a function of E/ρ and α, so that physical properties of the model depend only on α;

• in the limit $\alpha \to 0$ (and $\rho \to \infty$), the DOS exhibits a Landau levels spectrum for the average magnetic field with degeneracy $\rho\alpha$ by unit volume;

• for $\alpha = \frac{1}{2}$, the DOS is analytically shown to be a monotonically increasing function of E/ρ, without any Landau level. Moreover, it is characterized only by a depletion of states at the bottom of the spectrum of a Lifschitz type.

So we could infer the existence of a critical value α_c distinguishing between two regimes of the model:

• a Landau regime for $\alpha < \alpha_c$, characterized by Landau oscillations of the DOS where the mean magnetic field overbalances disorder. By Landau oscillations, we mean that the DOS is not monotonically increasing, and thus exhibits broadened Landau levels.

• a disordered regime for $\alpha > \alpha_c$, without Landau oscillations in the spectrum.

Analytical and numerical results showed that $\alpha_c > 0.29$, and a numerical investigation of the DOS gave $\alpha_c \simeq 0.35$.

In this paper, we consider the random magnetic impurities system as a gas of particles with a distribution obeying Fermi or Bose statistics at a temperature T_v. What we have in mind (especially in the Fermi case) is to study the effect of density correlations of the impurities on the DOS. With Fermi statistics, we aim to homogeneize the impurity configurations by varying T_v, considered here as an additional disorder parameter. We expect that this is going to affect the 'transition' described above. In our opinion, this approach for studying non-local correlations of impurities should be relevant for real disordered systems. In particular, we should expect, on a physical basis, impurities to repel each other, and a way to model this situation is precisely to consider impurities as a gas of fermions.

We will first properly define the perturbative expansion of the average partition function. Then, we will explicitly compute at order α^2 the contributions to the average partition function. At that order we can explicitly see the relative contribution of the mean magnetic field and its fluctuations, and have a first insight into the effect of density correlation on disorder. We will generalize these considerations at any order in α, for Fermi statistics at $T_v = 0$, to see how density correlations (repulsive for fermions) reduce the fluctuation contribution. Our main result will be that in the Fermi case, at zero temperature, the average DOS always displays Landau-like oscillations. Thus, for this type of impurity distribution, there is no transition between an isolated impurity-disordered regime and a mean magnetic field regime, contrarily to the Poissonian case.

2. The model

2.1. General formalism

Let us consider an electron coupled to a random magnetic field given by a distribution $\rho(\boldsymbol{r})$ of magnetic impurities. This means that $\rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$ is the number of impurities at position $\boldsymbol{r}$ in the infinitesimal volume $\mathrm{d}^2\boldsymbol{r}$. The Hamiltonian is given by

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \alpha \int \mathrm{d}^2\boldsymbol{r}'\,\rho(\boldsymbol{r}')\frac{\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\right)^2 \mp \frac{\alpha}{m}\rho(\boldsymbol{r}) \tag{1}$$

where we have explicitly taken into account the coupling of the magnetic field to the spin-up $(-)$ or down $(+)$ degree of freedom of the electron.

In the case of a discrete distribution $\rho(\boldsymbol{r}) = \sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i)$, i.e. points randomly dropped on the plane, where the index i indices the impurities, the spin term is a sum of contact terms. It corresponds to a choice of a peculiar self-adjoint extension [6]: in the spin-down case, the wavefunctions vanish at the location of the impurities (hard-core boundary condition), whereas in the spin-up case singular wavefunctions are considered at the location of the impurities (attractive-core conditions). In order to extract the short-distance behaviour of the wavefunctions, a non-unitary wavefunction redefinition has been used [7]:

$$\psi_N(\boldsymbol{r}) = \prod_{i=1}^{N} |\boldsymbol{r} - \boldsymbol{r}_i|^{\pm\alpha} \tilde{\psi}_N(\boldsymbol{r}). \tag{2}$$

The generalization of this transformation to the continuous distribution $\rho(\boldsymbol{r})$ introduced in (1) is

$$\psi(\boldsymbol{r}) = \mathrm{e}^{\pm\alpha \int \mathrm{d}^2\boldsymbol{r}' \rho(\boldsymbol{r}') \ln|\boldsymbol{r}-\boldsymbol{r}'|} \tilde{\psi}(\boldsymbol{r}). \tag{3}$$

The Hamiltonian $\tilde{H}$ acting on $\tilde{\psi}(\boldsymbol{r})$ rewrites

$$\tilde{H}_d = -\frac{2}{m}\partial_z\partial_{\bar{z}} - \frac{2\alpha}{m}\int \mathrm{d}z'\,\mathrm{d}\bar{z}'\,\frac{\rho(z', \bar{z}')}{\bar{z} - \bar{z}'}\partial_z \tag{4}$$

$$\tilde{H}_u = -\frac{2}{m}\partial_z\partial_{\bar{z}} + \frac{2\alpha}{m}\int \mathrm{d}z'\,\mathrm{d}\bar{z}'\,\frac{\rho(z', \bar{z}')}{z - z'}\partial_{\bar{z}} \tag{5}$$

where the complex coordinates in the plane have been used $z = x + \mathrm{i}y$, $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\mathrm{d}z\,\mathrm{d}\bar{z} = \mathrm{d}^2\boldsymbol{r}$. H or $\tilde{H}$ can be used indifferently to compute the partition function, since it is by definition the trace of a function of H. In the sequel we will concentrate on the spin-down coupling, keeping in mind that the spin-up analysis could be easily done following the same lines.

Until now the statistical properties of the distribution $\{\boldsymbol{r}_i\}$ in $\rho(\boldsymbol{r})$ have not been specified. If a Poissonian distribution is chosen, the statistical properties of $\rho(\boldsymbol{r})$ are defined by the cumulants†

$$\overline{\rho(\boldsymbol{r}_1)\ldots\rho(\boldsymbol{r}_k)} = \rho\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\delta(\boldsymbol{r}_2 - \boldsymbol{r}_3)\ldots\delta(\boldsymbol{r}_{k-1} - \boldsymbol{r}_k). \tag{6}$$

Here however, we deal with quantum statistics for the impurities themselves, so $\rho(\boldsymbol{r})$ has to be defined as an operator

$$\rho(\boldsymbol{r}) = \psi^+(\boldsymbol{r})\psi(\boldsymbol{r}) \tag{7}$$

with

$$\psi(\boldsymbol{r}) = \frac{1}{2\pi}\int \mathrm{d}^2\boldsymbol{k}\, a(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \tag{8}$$

$$\psi^+(\boldsymbol{r}) = \frac{1}{2\pi}\int \mathrm{d}^2\boldsymbol{k}\, a^+(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \tag{9}$$

$a^+(\boldsymbol{k})$ and $a(\boldsymbol{k})$ are the creation and annihilation Fock space particle operators of momentum $\boldsymbol{k}$, with the commutation rules

$$[a(\boldsymbol{k}), a^+(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}') \tag{10}$$

for bosons, and

$$\{a(\boldsymbol{k}), a^+(\boldsymbol{k}')\} = \delta(\boldsymbol{k} - \boldsymbol{k}') \tag{11}$$

† For a Gaussian magnetic field [2], all the cumulants vanish except the first and second, which has to be contrasted with the Poissonian cumulants given in (6).

for fermions.

Thus, the impurities, considered as a quantum gas, have a temperature T_v and chemical potential μ, which determines their mean density ρ. The average over disorder of an operator Q consists of

$$\langle Q\rangle = \frac{T_r[\mathrm{e}^{-\beta_v H_v} Q]}{T_r[\mathrm{e}^{-\beta_v H_v}]} \tag{12}$$

where the impurity second-quantized Hamiltonian

$$H_v = \int \mathrm{d}^2\boldsymbol{k}\, \left(\frac{k^2}{2m} - \mu\right) a^+(\boldsymbol{k}) a(\boldsymbol{k}) \tag{13}$$

describes the equilibrium state of the impurity gas in the grand-canonical ensemble. Note that we consider here quenched impurities, which are in thermodynamical equilibrium. Note also that the Poissonian distribution $\mathrm{d}P(\boldsymbol{r}_i) = \mathrm{d}\boldsymbol{r}_i/V$ can be seen as the particular case of Bose distribution at $T_v = 0$. The impurities indeed condensate in the zero-energy N-body wavefunction $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = (\frac{1}{\sqrt{V}})^N$, leading to the N-impurity Poissonian distribution $\mathrm{d}P \equiv \psi^*(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 \ldots \mathrm{d}\boldsymbol{r}_N = \prod_{i=1}^N \frac{\mathrm{d}\boldsymbol{r}_i}{V}$.

We wish to evaluate perturbatively the average one-electron partition function at inverse temperature β. From (4) one has

$$\langle \boldsymbol{r}|\mathrm{e}^{-\beta\tilde{H}}|\boldsymbol{r}\rangle = \sum_{p=0}^{\infty} \left(\frac{2\alpha}{m}\right)^p \int_0^\beta \mathrm{d}\beta_1 \ldots \int_0^{\beta_{p-1}} \mathrm{d}\beta_p\, G_{\beta-\beta_1}(\boldsymbol{r}, \boldsymbol{r}_1)\psi^+(\boldsymbol{r}'_1)\psi(\boldsymbol{r}'_1)$$
$$\times \frac{\partial_{z_1}}{\bar{z}_1 - \bar{z}'_1} G_{\beta_1-\beta_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) \ldots \psi^+(\boldsymbol{r}'_p)\psi(\boldsymbol{r}'_p) \frac{\partial_{z_p}}{\bar{z}_p - \bar{z}'_p} G_{\beta_p}(\boldsymbol{r}_p, \boldsymbol{r}) \tag{14}$$

where integrations over the position variables $\boldsymbol{r}_i$ and $\boldsymbol{r}'_i$ are implicit. $G_\beta(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is the free electron propagator

$$G_\beta(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{m}{2\pi\beta} \mathrm{e}^{-\frac{m}{2\beta}|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}. \tag{15}$$

Averaging over disorder yields expressions like

$$\langle \rho(\boldsymbol{r}'_1) \ldots \rho(\boldsymbol{r}'_p)\rangle = \frac{\mathrm{Tr}[\mathrm{e}^{-\beta_v H_v} \psi^+(\boldsymbol{r}'_1)\psi(\boldsymbol{r}'_1) \ldots \psi^+(\boldsymbol{r}'_p)\psi(\boldsymbol{r}'_p)]}{\mathrm{Tr}[\mathrm{e}^{-\beta_v H_v}]}$$

which can be evaluated using the contractions

$$\pm g_f^\pm(\boldsymbol{r}, \boldsymbol{r}') = \langle \psi(\boldsymbol{r})\psi^+(\boldsymbol{r}')\rangle = \int \frac{\mathrm{d}^2\boldsymbol{k}}{4\pi^2}(1 \pm n_{\boldsymbol{k}})\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \tag{16}$$

$$g_b^\pm(\boldsymbol{r}, \boldsymbol{r}') = \langle \psi^+(\boldsymbol{r})\psi(\boldsymbol{r}')\rangle = \int \frac{\mathrm{d}^2\boldsymbol{k}}{4\pi^2} n_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \tag{17}$$

$n_{\boldsymbol{k}}$ stands for the Bose–Einstein (upper sign) or Fermi–Dirac (lower sign) distributions.

$$n_{\boldsymbol{k}} = \frac{1}{\mathrm{e}^{\beta_v(k^2/2m-\mu)} \mp 1}. \tag{18}$$

One has the relation

$$g_f^\pm(\boldsymbol{r}, \boldsymbol{r}') = g_b^\pm(\boldsymbol{r}, \boldsymbol{r}') \pm \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{19}$$

To summarize, the perturbative expansion of the one-electron average partition function can be represented in terms of Feynmann diagrams given by rules analogous to those of a finite temperature second-quantized formalism

electron line: $G_\beta(\boldsymbol{r}_i, \boldsymbol{r}_j)$

forward impurity line: $g_f^{\pm}(\boldsymbol{r}'_i, \boldsymbol{r}'_{\sigma(i)})$
backward impurity line: $g_b^{\pm}(\boldsymbol{r}'_i, \boldsymbol{r}'_{\sigma(i)})$
impurity tadpole: $g_b^{\pm}(\boldsymbol{r}'_i, \boldsymbol{r}'_i) = \rho$
electron-impurity vertex: $\frac{2\alpha}{m} \frac{1}{\bar{z}_i - \bar{z}'_i} \partial_{z_i}$.

For a given diagram of order p, the electron propagates from its initial to its final position $\boldsymbol{r}$ via $\boldsymbol{r}_p, \boldsymbol{r}_{p-1}, \ldots, \boldsymbol{r}_1$, the location of electron-impurity interaction, with temperatures $0, \beta_p, \ldots, \beta_1, \beta$. The p vortices located at position $\boldsymbol{r}'_1 \ldots \boldsymbol{r}'_p$, undergo a permutation σ. At each $\sigma(i)$ corresponds a vortex line

backward line if $\sigma(i) > i$
forward line if $\sigma(i) < i$
and a tadpole if $\sigma(i) = i$.

In the Fermi case, each diagram is affected by the signature of the permutation σ.

The dimensionless parameters at work are the rescaled average impurity density $\lambda^2\rho$ in units of the electron thermal wavelength $\lambda^2 = 2\pi\beta/m$, the rescaled average density $\lambda_v^2\rho$ in units of the impurity thermal wavelength $\lambda_v^2 = 2\pi\beta_v/m$, and the Aharonov–Bohm coupling constant α.

Clearly, one expects that in the limit $\beta_v \to 0$, i.e. Boltzmann statistics, with uncorrelated randomly dropped impurities, one recovers the Poisson distribution. Also, as already advocated above, one expects that in the limit $\beta_v \to \infty$, in the Bose case, one has again the Poisson distribution, whereas the Fermi distribution leads to a less disordered situation. In the sequel, one will concentrate on the relative interplay between the dimensionless parameters $\lambda^2\rho$, $\lambda_v^2\rho$, α.

2.2. Mean-field expansion

Consider first diagrams that are entirely built by impurity tadpoles (figure 1(*a*)). These diagrams do not involve the many-body statistical correlations of the impurity distribution, and are thus independent of the statistics. Therefore, they yield the same contribution as in the Poisson case [4].

The order $\alpha^p (p > 0)$ term writes

$$Z_{\langle B \rangle}^{(p)} = -\frac{1}{\lambda^2} \frac{\zeta(1-p)}{(p-1)!} (-\lambda^2 \rho\alpha)^p.$$

Summation over p yields, as it should, the partition function per unit volume of the mean magnetic field $e\langle B \rangle = 2\pi\rho\alpha$ (i.e. in the mean-field limit $\alpha \to 0$, $\rho \to \infty$, $\rho\alpha$ finite)

$$Z_{\langle B \rangle} = \sum_{p=0}^{\infty} Z_{\langle B \rangle}^{(p)} = \frac{e\langle B \rangle}{2\pi} \frac{1}{2 \sinh \beta \frac{e\langle B \rangle}{2m}} \exp\left(-\beta \frac{e\langle B \rangle}{2m}\right) \qquad (20)$$

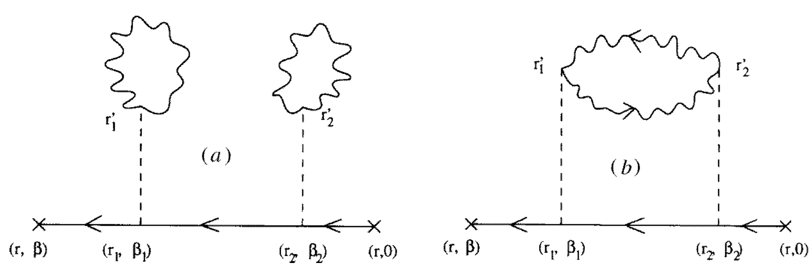

Figure 1. Diagrams contributing at second order: (*a*) mean-field diagram (*b*) disorder diagram.

where $Z^{(0)}_{\langle B\rangle} = 1/\lambda^2$ is the free partition function per unit volume. The positive shift in the Landau spectrum is a direct manifestation of the hard-core boundary conditions on the wavefunctions [4].

In the sequel, we will evaluate corrections to the mean-field partition function, which are responsible for mixing and broadening the Landau levels, and compare them with the Poissonian case.

2.3. Second-order expansion

To see the effect of fluctuations, we have to evaluate the partition function at least at second order in α, because the first order does not involve correlations of the magnetic field. Let us first recall the expression of the partition function up to second order in α for the Poissonian case [4]

$$\lambda^2 Z = 1 - \frac{1}{2}\lambda^2\rho\alpha + \frac{1}{2}(\lambda^2\rho)^2\alpha^2\left(\frac{1}{6} + \frac{1}{\lambda^2\rho}\right) + \cdots \tag{21}$$

which we can rewrite

$$Z = Z^{(0)}_{\langle B\rangle} + Z^{(1)}_{\langle B\rangle} + Z^{(2)}_{\langle B\rangle} + \tfrac{1}{2}\rho\alpha^2 + \cdots. \tag{22}$$

The last term on the right-hand side of (22) represents the first contribution due to fluctuations and vanishes like $1/\rho$ in the mean-field limit $\rho \to \infty, \alpha \to 0$, with $\alpha\rho$ kept fixed.

The corresponding term in the case of Bose or Fermi statistics is given by the diagram represented in figure 1(*b*). Its expression is

$$D(z) \equiv \pm\left(\frac{2\alpha}{m}\right)^2 \int_0^\beta \mathrm{d}\beta_1 \int_0^{\beta_1} \mathrm{d}\beta_2 \int \mathrm{d}^2\boldsymbol{r}_1\, \mathrm{d}^2\boldsymbol{r}_2\, \mathrm{d}^2\boldsymbol{r}'_1\, \mathrm{d}^2\boldsymbol{r}'_2 G_{\beta-\beta_1}(\boldsymbol{r},\boldsymbol{r}_1) g_b(\boldsymbol{r}'_1,\boldsymbol{r}'_2)$$
$$\times \frac{\partial_{z_1}}{\bar{z}_1 - \bar{z}'_1} G_{\beta_1-\beta_2}(\boldsymbol{r}_1,\boldsymbol{r}_2) g_f(\boldsymbol{r}'_2,\boldsymbol{r}'_1) \frac{\partial_{z_2}}{\bar{z}_2 - \bar{z}'_2} G_{\beta_2}(\boldsymbol{r}_2,\boldsymbol{r})$$

where z is the fugacity $z = \mp \mathrm{e}^{\beta_v\mu}$. In the interval $z \in]-1, 1]$ one has the expansion

$$n(\boldsymbol{k}) = \pm\sum_{n=1}^{\infty}(-z)^n \mathrm{e}^{-n\beta_v\frac{k^2}{2m}}. \tag{23}$$

As a result

$$g_f(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') + \sum_{n=1}^{\infty}(-z)^n G_{n\beta_v}(\boldsymbol{r},\boldsymbol{r}') \tag{24}$$

$$g_b(\boldsymbol{r},\boldsymbol{r}') = \pm\sum_{n=1}^{\infty}(-z)^n G_{n\beta_v}(\boldsymbol{r},\boldsymbol{r}'). \tag{25}$$

The density ρ is related to z by

$$\rho = \mp\frac{1}{\lambda_v^2}\ln(1+z). \tag{26}$$

Using (24), (25), the contribution $D(z)$ of the diagram of figure 1(*b*) is

$$D(z) = \frac{\rho\alpha^2}{2} - \frac{m\alpha^2}{4\beta_v}\sum_{\substack{n=1\\p=1}}^{\infty}\frac{(-z)^{n+p}}{n+p}\left(1 - \int_0^1 \mathrm{d}x\,\frac{x_{np}}{x_{np} + x(1-x)}\right) \tag{27}$$

where $x_{np} = \frac{np}{n+p}\frac{\beta_v}{\beta}$.

At high temperature ($T_v > T$), in the Boltzmann limit, i.e. $z \to 0$ in both the Bose and Fermi cases, one gets $D(z) \to \rho\alpha^2/2$, which is precisely the Poisson distribution result. The correction reads

$$D(z) = \frac{\rho\alpha^2}{2}\left(1 - \frac{\pi}{2}\lambda_v^2\rho\right) + \cdots . \tag{28}$$

Let us now concentrate on the low-temperature limit ($T_v < T, x_{np} \to +\infty$). In that case, $D(z)$ rewrites as

$$D(z) = \frac{\rho\alpha^2}{2}\left[1 \mp \sum_{q=1}^{\infty}\sum_{m=0}^{q-1}(-\lambda^2\rho)^q \frac{(q!)^2}{(2q+1)!} C_{q-1}^m \frac{h_{q-m}(z)h_{m+1}(z)}{[\ln(1+z)]^{q+1}}\right] \tag{29}$$

where the functions $h_q(z)$'s are defined in the interval $]-1, 1]$ by

$$h_q(z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{n^q} \tag{30}$$

and the C_{q-1}^m's are the binomial coefficients. The low-temperature limit corresponds to $z \to -1$ in the Bose case and $z \to +\infty$ in the Fermi case. It happens that (30), and therefore (29) can be analytically continuated in $]-1, +\infty[$ by noticing that

$$h_0(z) = -\frac{z}{1+z} \tag{31}$$

and by using the recursive relation

$$h_{q+1}(z) = \int_0^z \frac{h_q(x)}{x}\, \mathrm{d}x. \tag{32}$$

It follows that the expansion

$$h_q(z) = -\frac{(\ln z)^q}{q!} + 2\sum_{n=1}^{E(q/2)} h_{2n}(1)\frac{(\ln z)^{q-2n}}{(q-2n)!} - (-1)^q \sum_{n=1}^{\infty}\frac{(-1)^n}{n^q}\frac{1}{z^n} \tag{33}$$

is valid for $z > 1$.

Which contributions does $D(z)$ yield in the low T_v limit? In the Bose case, some care is required, since in (29) the limit $z \to 0$ cannot be interchanged with summations. We checked numerically that (29) indeed yields $\rho\alpha^2/2$, i.e. the Poissonian result as expected. In the Fermi case, (29), (33) yield

$$\lim_{z\to+\infty} D(z) = \frac{\rho\alpha^2}{2}\left[1 - \frac{1}{\lambda^2\rho}\left(1 + \lambda^2\rho - \mathrm{e}^{-\lambda^2\rho} - 2\sqrt{\lambda^2\rho}\int_0^{\sqrt{\lambda^2\rho}} \mathrm{d}y\, \mathrm{e}^{-y^2}\right)\right]. \tag{34}$$

At this point one can consider either a low-impurity density limit $\lambda^2\rho \ll 1$, or a high-impurity density limit $\lambda^2\rho \gg 1$. At low density $\lambda^2\rho << 1$, one finds that the average partition function per unit volume rewrites as

$$\lambda^2 Z = 1 - \frac{1}{2}\lambda^2\rho\alpha + \frac{1}{2}(\lambda^2\rho)^2\alpha^2\left(\frac{1}{\lambda^2\rho} + \frac{1}{30}\lambda^2\rho + \cdots\right) + \cdots . \tag{35}$$

Comparing with (21), we see that there is no $\rho^2\alpha^2$ term, a situation quite different from the Poissonian case, where this term is precisely the leading mean magnetic field term. We will come back to this point later.

On the other hand, to see how the system reaches the mean-field limit, we have to use the high-density expansion ($\lambda^2\rho \gg 1$). Equation (34) leads to

$$Z = Z_{\langle B\rangle}^{(0)} + Z_{\langle B\rangle}^{(1)} + Z_{\langle B\rangle}^{(2)} + \frac{1}{2}\rho\alpha^2\left(\sqrt{\frac{\pi}{\lambda^2\rho}} - \frac{1}{\lambda^2\rho} + \cdots\right) + \cdots \tag{36}$$

where the corrections due to fluctuations at second order in α are explicitly given.

3. Fermi case at zero temperature: The Landau regime

3.1. The ordered regime

Before the system reaches the mean-field limit (20), we expect [4] an intermediate regime characterized by smooth Landau oscillations in the spectrum. This intermediate regime is identified as ordered by opposition to the one with no oscillation (disordered regime). The order α^2 is quite instructive to give information on the way the system reaches the mean-field limit. Consider indeed the case of Poissonian impurities (22). We observe $1/\rho$ corrections to the mean field at any order of the perturbative expansion in $(\alpha\rho)^n$. On the other hand, (36) shows corrections to the mean field of order $1/(\rho\sqrt{\rho})$. This implies that the system approaches more rapidly its mean-field limit when the impurities are fermions at zero temperature, rather than Poissonian. In other words, the system is less disordered, since a Fermi distribution of impurities is more homogeneous than a Poissonian one.

Let us generalize these considerations at any order of perturbative theory in α. One has to evaluate $\langle\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\ldots\rho(\boldsymbol{r}_n)\rangle$, which can be rewritten as

$$\langle\rho(\boldsymbol{r}_1)..\rho(\boldsymbol{r}_n)\rangle = \sum_{p=1}^{n}\sum_{f\in S_n^p}\frac{1}{p!}\int \mathrm{d}\boldsymbol{r}'_1\ldots\mathrm{d}\boldsymbol{r}'_p\,\rho(\boldsymbol{r}'_1,\ldots,\boldsymbol{r}'_p)\prod_{q=1}^{n}\delta(\boldsymbol{r}_q-\boldsymbol{r}'_{f(q)}). \tag{37}$$

S_n^p is the set of all possible surjections from $(1,\ldots,n)$ to $(1,\ldots,p)$ and $\rho(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_p)$ is the p-body correlation function

$$\rho(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_p) = \sum_{\sigma\in S_p}\epsilon(\sigma)g_b(\boldsymbol{r}_1-\boldsymbol{r}_{\sigma(1)})\ldots g_b(\boldsymbol{r}_p-\boldsymbol{r}_{\sigma(p)}). \tag{38}$$

In the case of fermions at zero temperature, one has the correlator

$$g_b(\boldsymbol{r}) = \frac{1}{r}\sqrt{\frac{\rho}{\pi}}J_1\left(\sqrt{4\pi\rho}r\right). \tag{39}$$

For example, using

$$\langle\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\rangle = \rho^2 - \frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^2}\frac{\rho}{\pi}\left[J_1\left(\sqrt{4\pi\rho}|\boldsymbol{r}_1-\boldsymbol{r}_2|\right)\right]^2 + \rho\delta(\boldsymbol{r}_1-\boldsymbol{r}_2) \tag{40}$$

together with (14), yields contibution (34), in addition to the mean-field term.

At high-impurity density ($\lambda^2\rho\gg 1$), because of Fermi exclusion, the p-body correlation function becomes

$$\rho(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_p) = \rho^p\left[1-\frac{1}{\rho}\sum_{i<j}\delta(\boldsymbol{r}_i-\boldsymbol{r}_j)+\mathrm{O}\left(\frac{1}{\rho\sqrt{\rho}}\right)\right]. \tag{41}$$

When (41) is used for evaluating $\langle\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\ldots\rho(\boldsymbol{r}_n)\rangle$, one indeed finds that corrections to the mean-field term $(\rho\alpha)^n$ are of order $1/(\rho\sqrt{\rho})$.

3.2. Absence of a pure disordered regime

We have just seen that corrections to the average magnetic field limit are less important in the Fermi case. Could it be that the Fermi statistics of the impurities alter the occurrence of the transition itself [4]? We are precisely going to show that, at zero temperature, the DOS always exhibits oscillations in the whole range of the definition of $\alpha\in[0,\frac{1}{2}]$.

First, let us remark that the average partition function can be expressed as

$$\lambda^2 Z = F(\lambda^2\rho,\lambda_v^2\rho,\alpha) \tag{42}$$

which means that the average DOS is a function of E/ρ, $\lambda_v^2\rho$ and α. This is due to the fact that $g_b(\boldsymbol{r})$ is ρ times a function of $\sqrt{\rho}r$ and $\lambda_v^2\rho$. In (14) together with (37), rescaling β_i into β_i/β, $\boldsymbol{r}_i$ into $\boldsymbol{r}_i/\lambda$ and $\boldsymbol{r}'_i$ into $\sqrt{\rho}\boldsymbol{r}'_i$, immediately leads to (42). In particular at $T_v = 0$, $\lambda^2 Z$ is a function only of $\lambda^2\rho$ and α. As a consequence, at $T_v = 0$, we can expand $\lambda^2 Z$ in powers of $\lambda^2\rho = (2\pi\beta/m)\rho$

$$\lambda^2 Z = 1 + \tfrac{1}{2}\alpha(\alpha-1)\lambda^2\rho + c_2(\alpha)(\lambda^2\rho)^2 + c_3(\alpha)(\lambda^2\rho)^3 + \cdots. \tag{43}$$

The striking feature for fermion at $T_v = 0$ is that $c_2(\alpha)$ can be completely evaluated and found to vanish, a generalization of the observation made in (35). To see that, use simply

$$g_b(\boldsymbol{r}) = \rho \sum_{k=0}^{\infty} \frac{(-\pi\rho r^2)^k}{k!(k+1)!} \tag{44}$$

and conclude in general that $\rho(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_p)$ is at least of order ρ^p. In particular

$$\rho(\boldsymbol{r}_1, \boldsymbol{r}_2) = \rho^2 - [g_b(\boldsymbol{r}_1 - \boldsymbol{r}_2)]^2 \tag{45}$$

starts at order ρ^3, whereas $\rho(\boldsymbol{r}) = \rho$. Therefore, no contribution of order ρ^2 is to be found in $\langle\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\ldots\rho(\boldsymbol{r}_n)\rangle$, which means that the average partition function does not contain terms of order ρ^2 at any order in α.

Now in order to test oscillations in the DOS, we develop an argument given in [4], based on the specific heat

$$C = k\beta^2 \frac{\mathrm{d}^2}{\mathrm{d}\beta^2} \ln Z. \tag{46}$$

Using the basic definition of Z and integrations by parts in (46), we can show, in the small β limit, that

$$C - C_0 = 2\pi^2 k\beta^2 \int_0^\infty \int_0^\infty \mathrm{d}E\,\mathrm{d}E' \frac{\mathrm{d}\langle\rho(E)/V\rangle}{\mathrm{d}E} \frac{\mathrm{d}\langle\rho(E')/V\rangle}{\mathrm{d}E'} (E - E')^2 + \cdots \tag{47}$$

where $C_0 (= k)$ is the free specific heat. Thus, $(C - C_0)$ is positive if the DOS grows monotonically.

On the other hand, the small β expansion (43), taking into account $c_2(\alpha) = 0$, leads to

$$C - C_0 = -\frac{1}{4} k \left(\frac{2\pi\beta\rho}{m}\right)^2 \alpha^2(1-\alpha)^2 + \cdots \tag{48}$$

which is negative. Therefore the average DOS always displays Landau oscillations when the magnetic impurities obey Fermi statistics at zero temperature.

4. Conclusion

To conclude this analysis, let us emphasize again that for intermediate magnetic impurity temperature, the average partition function has been shown in (42) to scale as $1/\lambda^2$ times a function of $\lambda^2\rho$, $\lambda_v^2\rho$ and α. This implies that the average DOS is a function $\langle\rho(E/\rho, \lambda_v^2\rho, \alpha)\rangle$. Since the Poisson distribution is recovered in the Boltzmann limit $T_v \to \infty$ (since then $\rho(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_p) \to \rho^p$), one interpolates between the Poisson and zero-temperature Fermi cases simply by varying $\lambda_v^2\rho$ from 0 to ∞. When $\lambda_v^2\rho = 0$, there is a transition at $\alpha_c \simeq 0.35$, whereas when $\lambda_v^2\rho = \infty$, we have just shown that no transition occurs at all. Therefore, we expect that, for $\lambda_v^2\rho$ sufficiently small, a transition still occurs at a critical value $\alpha_c(\lambda_v^2\rho) > 0.35$, and for $\lambda_v^2\rho$ sufficiently big, no transition occurs any more, meaning that the system is always Landau like, in the whole interval $\alpha \in [0, \frac{1}{2}]$ (figure 2). It is not clear whether the transition observed can be interpreted as a phase transition.

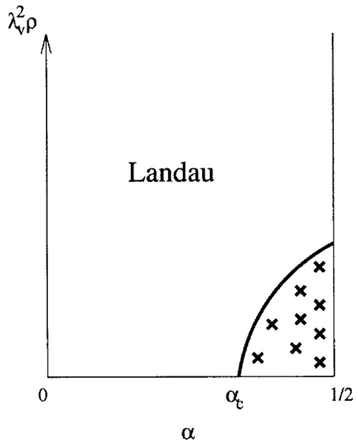

Figure 2. 'Phase diagram' separating the Landau regime from the disordered regime. The heavy curve indicates the crossover between an oscillating and monotonically increasing DOS.

But magnetic impurity distributions do influence this transition by actually re-ordering the system when the impurity density correlations are increased.

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